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Feynman rules for non-equilibrium field theory

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Abstract. Within the closed-time-path formalism, a set of real- and imaginary-time propagators is obtained, which are suitable for the perturbative evaluation of Green functions in non-equilibrium states of a scalar field theory. A generalized renormalization procedure, described in earlier works, allows for the partial resummation of absorptive parts of loop diagrams, so that finite quasiparticle lifetimes are incorporated in the unperturbed propagators. In this way, low-order calculations are made to reflect the time evolution of the non-equilibrium state, which is not correctly described by standard perturbation theory.

1. Introduction

It is often important to know how the state of a physical system described by a quantum field theory changes in response to a changing environment. In particular, when the system undergoes a phase transition, large departures from thermal equilibrium may be expected, and special methods are required to describe the dynamics of such events. The situations we have particularly in view are the phase transitions which may have occurred in the early, rapidly expanding universe, with profound cosmological consequences (see, for example, the works cited in [1]). However, the techniques needed to deal satisfactorily with this problem should be much more generally applicable.

As is well known, systems which remain in thermal equilibrium can be described by the imaginary-time formalism [2], which exploits the formal analogy between the statistical density operator $\rho = \exp(-\beta H)$ and the time evolution operator $\exp(-iHt)$. Real-time information can, in principle, be obtained by analytic continuation of the Green functions to real times, although this continuation is not always straightforward [3]. For non-equilibrium states, and more particularly for systems with time-dependent Hamiltonians, it is essential to formulate the theory in real time. The theoretical framework for doing this, known as the closed-time-path formalism, was first studied by Schwinger and Keldysh [4] and has since been developed by many others. This formalism is reviewed in [5].

In this paper, we are concerned with systems whose Hamiltonians depend explicitly on time, but whose state at an initial time, say $t = 0$, is assumed to be one of thermal equilibrium. For scalar field theories of this type, an elegant path-integral version of the closed-time-path formalism was described several years ago by Semenoff and Weiss [6]. Physically the assumption of an initial equilibrium state

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sits uneasily with a time-dependent Hamiltonian, but it may be quite reasonable under appropriate circumstances. For example, the spectrum of black-body radiation in an expanding universe is well known to remain thermal, but with a temperature inversely proportional to the scale factor. Mathematically, this assumption is extremely advantageous, since it leads to a cleanly defined problem in field theory. Once this problem is sufficiently well understood, it may well be possible to consider more general initial states. Indeed, steps in this direction have been taken by Calzetta and Hu [7] in the context of a model with a time-independent Hamiltonian.

In the formalism of Semenoff and Weiss, each quantum field ϕ is represented by a set of three path-integration variables $\{\phi_1, \phi_2, \phi_3\}$. They inhabit the three segments of a contour in the complex time plane which runs from $t = 0$ along the real axis to a final time T, returns along the real axis to $t = 0$ and finally descends to $t = -i\beta_0$, where β_0 is the inverse of the initial temperature. Thus, for example, a Heisenberg-picture operator $A(t)$ which is represented in the Schrödinger picture by A_S is given by $A(t) = U^{-1}(t)A_S U(t)$, where $U(t)$ is the time-evolution operator. Its expectation value is

$$\langle A(t)\rangle = \mathrm{Tr}[\rho U^{-1}(t)A_S U(t)] \tag{1.1}$$

and the three segments of the contour arise from the path-integral representations of the operators $U(t)$, $U^{-1}(t)$ and ρ. More generally, a generating functional for Green functions with arbitrary real-time ordering and for imaginary-time Green functions can be constructed by including independent source terms in these three operators.

Semenoff and Weiss derived Feynman rules for the perturbative evaluation of the Green functions, but these rules present an awkward problem in practice. Organized in the obvious way, perturbation theory is an expansion about a free-particle theory. In this context, it makes approximate sense to refer to single-quasiparticle modes of excitation, and the unperturbed propagators involve the occupation numbers of these modes. These occupation numbers ought to evolve with time, to reflect the changing state of the system. However, the dissipative relaxation processes which give rise to this evolution necessarily involve scattering, which is absent from the unperturbed theory. Thus, the unperturbed propagators always involve the occupation numbers appropriate to the initial time. Since only low-order calculations are tractable in practice, one cannot expect to obtain expectation values, say, which evolve with time in the correct manner.

In two earlier papers [8, 9], we showed how perturbation theory may be reorganized so as to remedy this situation. The idea, reviewed in section 2, is to effect an approximate resummation of the absorptive parts of higher-order contributions to the full propagators, incorporating these into modified versions of the unperturbed propagators. One then finds that the quasiparticle modes of the unperturbed theory have finite decay widths, and the associated relaxation times characterize the evolution of their occupation numbers. In the spirit of standard renormalization theory, this can be achieved by adding a suitable counterterm to the unperturbed part of the action and subtracting it from the interaction part. By choosing the counterterm in such a way that the new interaction vertex cancels (as nearly as possible) the higher-order contributions to the self-energy, one optimizes the unperturbed propagator as an approximation to the full propagator. When the time evolution is sufficiently slow, and the decay widths sufficiently small, we found that the occupation numbers approximately obey a kinetic equation of the Boltzmann type—a result which in some measure confirms the efficacy of our approach to the problem.

Our attention in these papers was restricted to the 2×2 matrix of real-time propagators. To check that the technique is fully consistent, it is necessary to show that this matrix can be embedded in the 3×3 matrix of real- and imaginary-time propagators required by the complete formalism, and that is the purpose of the present work. It was pointed out in [8, 9] that in practice, only the real-time propagators are needed for low-order calculations, provided that one inserts the initial occupation numbers by hand. It is nevertheless desirable to assure ourselves that the whole formalism can be consistently constructed. Moreover, when the theory is extended to incorporate gauge fields (a task which will be discussed in a separate publication) the propagators have a somewhat complicated, gauge-dependent structure. Only by carefully constructing the full 3×3 propagator matrix, as was done by Kobes *et al* [10] for the case of thermal equilibrium, can the correct results be obtained.

It turns out that the strategy needed to complete the construction of the propagator matrix satisfactorily is a slightly curious one (or so it seems to the author), and for this reason we present our analysis in some detail. The theory of the real-time propagators is reviewed in section 2. In section 3, we derive the boundary conditions satisfied by the nine real- and imaginary-time propagators and obtain expressions for these. Though formally correct, these propagators are unsatisfactory, since they do not properly reflect the initial equilibrium state. We explain why this comes about and describe the remedy. Finally, our results are summarized and discussed in section 4.

2. Real-time propagators

We study the theory of a real scalar field $\phi(\boldsymbol{x}, t)$, defined by the action

$$S(\phi) = \int_0^T \mathrm{d}t \int \mathrm{d}^3x \left[\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2(t)\phi^2 - \frac{\lambda}{4!}\phi^4\right] \tag{2.1}$$

which, with a suitable choice of the time-dependent mass $m(t)$, would be appropriate for a Robertson–Walker universe. Using standard path-integral methods [6, 11], one finds that the generating functional for real- and imaginary-time Green functions is given by

$$Z(J_1, J_2, J_3) = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \mathcal{D}\phi_3 \exp\left[\mathrm{i}\bar{S}(\phi_1, \phi_2, \phi_3) + \mathrm{i}\oint \mathrm{d}t\, \mathrm{d}^3x\, J \cdot \phi\right] \tag{2.2}$$

where

$$\bar{S}(\phi_1, \phi_2, \phi_3) = S(\phi_1) - S(\phi_2) + \mathrm{i}S_{\mathrm{E}}(\phi_3) \tag{2.3}$$

S_{E} denotes the Euclidean action

$$S_{\mathrm{E}}(\phi) = \int_0^{\beta_0} \mathrm{d}\tau \int \mathrm{d}^3x \left[\frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2(0)\phi^2 + \frac{\lambda}{4!}\phi^4\right] \tag{2.4}$$

and the time integration means

$$\oint \mathrm{d}t\, J \cdot \phi = \int_0^T \mathrm{d}t\,(J_1\phi_1 + J_2\phi_2) + \int_0^{\beta_0} \mathrm{d}\tau\, J_3\phi_3. \tag{2.5}$$

The full propagators are given by

$$G_{ab}(x;x') = -\frac{\delta}{\delta J_a(x)}\frac{\delta}{\delta J_b(x')} \ln Z(J_1, J_2, J_3)\Big|_{J_1=J_2=J_3=0} \tag{2.6}$$

where x denotes $(\boldsymbol{x}, t)$ for $a, b = 1, 2$ or $(\boldsymbol{x}, \tau)$ for $a, b = 3$. In terms of the original quantum field, the real-time propagators $(a, b = 1, 2)$ are the expectation values

$$\begin{pmatrix} G_{11}(x;x') & G_{12}(x;x') \\ G_{21}(x;x') & G_{22}(x;x') \end{pmatrix} = \mathrm{Tr}\left[\rho \begin{pmatrix} T[\phi(x)\phi(x')] & [\phi(x')\phi(x)] \\ [\phi(x)\phi(x')] & \bar{T}[\phi(x)\phi(x')] \end{pmatrix}\right] \tag{2.7}$$

where T and $\bar{T}$ denote time- and anti-time-ordering respectively.

We showed in [9] that these propagators can be expressed in the form

$$\mathbf{G}(x;x') = \theta(t-t')\begin{pmatrix} H(x;x') & H^*(x;x') \\ H(x;x') & H^*(x;x') \end{pmatrix} + \theta(t'-t)\begin{pmatrix} H(x';x) & H(x';x) \\ H^*(x';x) & H^*(x';x) \end{pmatrix} \tag{2.8}$$

so that they are all determined by a single complex function $H(x;x')$. Since we are dealing with a spatially homogeneous system, the spatial arguments appear only in the combination $(\boldsymbol{x} - \boldsymbol{x}')$, and it is convenient to take a Fourier transform on this variable:

$$H(x;x') = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')] H_k(t,t'). \tag{2.9}$$

Perturbation theory requires a lowest-order approximation to the propagators, whose real-time components we denote by $\mathbf{g}_k(t,t')$, and these can be constructed from a function $h_k(t,t')$ as in (2.8). They are determined in the usual way as solutions of an equation of the form

$$\mathcal{D}_k(t, \partial/\partial t)\mathbf{g}_k(t,t') = \mathbf{g}_k(t,t')\mathcal{D}_k(t', \overleftarrow{\partial/\partial t'}) = -\mathrm{i}\delta(t-t')\mathbf{1} \tag{2.10}$$

where $\mathcal{D}_k$ is a differential operator associated with the unperturbed part of the action (2.3). If we were to choose this unperturbed action simply by taking the quadratic part of (2.1) (and ignoring, for now, the imaginary-time contribution $S_{\mathrm{E}}(\phi_3)$), then this differential operator would be

$$\mathcal{D}_k = \begin{pmatrix} \partial^2/\partial t^2 + k^2 + m^2(t) & 0 \\ 0 & -\partial^2/\partial t^2 - k^2 - m^2(t) \end{pmatrix}. \tag{2.11}$$

However, in order to incorporate dissipative effects, it is helpful to choose a different unperturbed action, and thus obtain a different form for $\mathcal{D}_k$. The idea is to optimize $\mathbf{g}$ as an approximation to $\mathbf{G}$ by adding to $\mathcal{D}_k$ terms which mimic the effect of higher-order contributions to the self-energy. In order that it be possible to solve (2.10), we require $\mathcal{D}_k$ to be a local, second-order differential operator. We find [9] that the most general form of this operator consistent with the structure of the real-time self-energy matrix is

$$\mathcal{D}_k = \begin{pmatrix} \partial^2/\partial t^2 + \beta_k(t) - \mathrm{i}\alpha_k(t) & \gamma_k(t)\partial/\partial t + \frac{1}{2}\dot{\gamma}_k(t) + \mathrm{i}\alpha_k \\ -\gamma_k(t)\partial/\partial t - \frac{1}{2}\dot{\gamma}_k(t) + \mathrm{i}\alpha_k(t) & -\partial^2/\partial t^2 - \beta_k(t) - \mathrm{i}\alpha_k(t) \end{pmatrix} \tag{2.12}$$

where $\alpha_k(t)$, $\beta_k(t)$ and $\gamma_k(t)$ are real functions, even in k, which are yet to be determined.

This differential operator arises from the action

$$\tilde{S}_0(\phi_1,\phi_2,\phi_3) = \frac{1}{2}\int dt \int \frac{d^3k}{(2\pi)^3}\,[\dot{\phi}_{1k}\dot{\phi}_{1-k} - \dot{\phi}_{2k}\dot{\phi}_{2-k} - \beta_k(\phi_{1k}\phi_{1-k} - \phi_{2k}\phi_{2-k}) - \gamma_k(\phi_{1k}\dot{\phi}_{2-k} - \phi_{2k}\dot{\phi}_{1-k}) + i\alpha_k(\phi_{1k} - \phi_{2k})(\phi_{1-k} - \phi_{2-k})] + \cdots \tag{2.13}$$

where the ellipsis represents the ϕ_3 contribution which does not yet concern us. The interaction part of the action is $\tilde{S} - \tilde{S}_0$, and it contains a counterterm vertex involving α_k, β_k and γ_k. Thus, these functions may be chosen in such a way that the counterterm vertex cancels some part of the higher-order contributions to the self-energy. In this way, the higher-order contributions are partially resummed, to appear in the unperturbed propagators **g**.

It is perhaps worth emphasizing that the functions α_k, β_k and γ_k do not represent an arbitrary modification of our original theory; they merely represent an arbitrary choice of the lowest-order theory about which we perturb and this choice can be made so as to optimize our low-order approximations. It will be seen, however, that the action $\tilde{S}_0$ in (2.13) has a structure different from that of (2.3), which consists of a sum of terms, each depending on only one of the fields ϕ_1, ϕ_2 and ϕ_3. Since the latter structure arises automatically in the path-integral representation of a quantity such as (1.1), our lowest-order theory is not in itself a true quantum field theory. Indeed, the dissipative behaviour we are trying to describe arises mathematically only when the theory is treated in some approximate manner, for example, by truncating the perturbation series at some finite order. This point is discussed in some detail by Calzetta and Hu [7]. On the other hand, the unitarity of the time-evolution operator $U(t)$ and the Hermiticity of the density operator ρ are reflected in a symmetry of $\tilde{S}$, which reads

$$\tilde{S}^*(\phi_1,\phi_2,\phi_3) = -\tilde{S}(\phi_2,\phi_1,\tilde{\phi}_3) \tag{2.14}$$

where

$$\tilde{\phi}_3(\boldsymbol{x},\tau) = \phi_3(\boldsymbol{x},\beta-\tau) \tag{2.15}$$

and we see that the unperturbed action (2.13) is consistent with this symmetry. (Note that, since $\phi_a(\boldsymbol{x},t)$ is real, $\phi^*_{ak}(t) = \phi_{a-k}(t)$.)

Given the differential operator (2.12), we can formally solve (2.10) for the real-time propagators **g**. The function h_k from which they are constructed in the form (2.8) is given, in terms of two auxiliary functions $\Omega_k(t)$ and $N_k(t)$ by [9]

$$h_k(t,t') = \tfrac{1}{4}\left[\Omega_k(t)\Omega_k(t')\right]^{-1/2}\exp\left(-\tfrac{1}{2}\gamma_k(t',t)\right) \times \left\{\left[1+N_k(t')\right]\exp\left(-i\Omega_k(t',t)\right) + \left[-1+N_k^*(t')\right]\exp\left(i\Omega_k(t',t)\right)\right\} \tag{2.16}$$

where

$$\gamma_k(t',t) = \int_{t'}^{t}\gamma_k(t'')\,dt''$$

and $\Omega_k(t',t)$ is similarly defined. The time-dependent frequency $\Omega_k(t)$ is any solution of the equation

$$\frac{1}{2}\frac{\ddot{\Omega}_k}{\Omega_k} - \frac{3}{4}\frac{\dot{\Omega}_k^2}{\Omega_k^2} + \Omega_k^2 = \beta_k - \tfrac{1}{4}\gamma_k^2 \tag{2.17}$$

while $N_k(t)$ satisfies

$$\left(\frac{\partial}{\partial t} + \gamma_k + 2i\Omega_k - \frac{\dot{\Omega}_k}{\Omega_k}\right)\left(\frac{\partial}{\partial t} + \gamma_k\right) N_k = 2i\alpha_k . \tag{2.18}$$

In view of the θ functions appearing in (2.8), the factor $\exp[-\frac{1}{2}\gamma_k(t',t)]$ always represents a decaying mode, so long as $\gamma_k(t)$ is positive.

From now on, we simplify our notation by suppressing the wavevector subscripts. The solution to (2.18) may be written as

$$N(t) = e^{-\gamma(0,t)}\left[N_1 + N_2 e^{-2i\Omega(0,t)}\right] + \int_0^t dt'\, e^{-\gamma(t',t)}\left[1 - e^{-2i\Omega(t',t)}\right]\frac{\alpha(t')}{\Omega(t')} \tag{2.19}$$

where N_1 and N_2 are (k-dependent) constants of integration, and N_1 is real. Semenoff and Weiss [6] in effect determine these constants (for the case $\alpha_k(t) = \gamma_k(t) = 0$) by considering the whole (3×3) propagator matrix and applying suitable boundary conditions at the endpoints of the three segments of the time contour. These boundary conditions will be discussed in detail in the following section. At this point, we merely record the values which are obtained when our dissipative counterterm is omitted; that is to say, when $\alpha(t) = \gamma(t) = 0$ and $\beta(t) = m^2(t) + k^2$. Assuming that the imaginary-time contribution to $\bar{S}_0$ (equation (2.13)) is just the quadratic part of (2.4), and taking a solution to (2.17) which satisfies $\Omega(t=0) = \omega \equiv +\sqrt{(m^2(0) + k^2)}$, we obtain

$$N_1 = \left[1 + \frac{\dot{\Omega}^2(0)}{8\omega^4}\right]\coth\left(\frac{1}{2}\beta_0\omega\right) \tag{2.20}$$

and

$$N_2 = \frac{1}{2\omega^2}\left[i\dot{\Omega}(0) - \frac{\dot{\Omega}^2(0)}{4\omega^2}\right]\coth\left(\frac{1}{2}\beta_0\omega\right). \tag{2.21}$$

We observe that the real-time propagators involve the mode functions

$$f_\pm(t) = (2\Omega(t))^{-1/2}\exp[\pm i\Omega(0,t)] \tag{2.22}$$

which correctly incorporate the time-dependent frequency, and the occupation numbers

$$n(t) = [\exp(\beta_0\omega) - 1]^{-1} = \tfrac{1}{2}\left[\coth\left(\tfrac{1}{2}\beta_0\omega\right) - 1\right] \tag{2.23}$$

which, incorrectly, are fixed at their initial values. In the case of a system which remains in thermal equilibrium, with $\dot{\Omega}(t) = 0$, we have just $N_1 = 2n + 1$ and $N_2 = 0$.

For the purpose of carrying out low-order calculations in the non-equilibrium theory, involving only the real-time propagators, it is probably sufficient to insert suitable values of N_1 and N_2 into (2.19) by hand. To derive these constants correctly, however, it is necessary to construct the complete propagator matrix. That is the central task of this paper and is addressed in the following section.

3. Imaginary- and mixed-time propagators

From the derivation of the path integral (2.2), it is apparent that the full propagators $G(x;x')$ satisfy the continuity conditions

$$\begin{aligned} G_{a1}(\boldsymbol{x},t;\boldsymbol{x}',T) &= G_{a2}(\boldsymbol{x},t;\boldsymbol{x}',T) \\ G_{a2}(\boldsymbol{x},t;\boldsymbol{x}',0) &= G_{a3}(\boldsymbol{x},t;\boldsymbol{x}',0) \\ G_{a1}(\boldsymbol{x},t;\boldsymbol{x}',0) &= G_{a3}(\boldsymbol{x},t;\boldsymbol{x}',\beta_0) \end{aligned} \tag{3.1}$$

($a = 1,2,3$). Using the Heisenberg equations of motion for the original field operators, it is also straightforward to derive the conditions [6]

$$\begin{aligned} \frac{\partial}{\partial t'}G_{a1}(x;x')\Big|_{t'=T} &= \frac{\partial}{\partial t'}G_{a2}(x;x')\Big|_{t'=T} \\ \frac{\partial}{\partial t'}G_{a2}(x;x')\Big|_{t'=0} &= \mathrm{i}\frac{\partial}{\partial \tau'}G_{a3}(x;x')\Big|_{\tau'=0} \\ \frac{\partial}{\partial t'}G_{a1}(x;x')\Big|_{t'=0} &= \mathrm{i}\frac{\partial}{\partial \tau'}G_{a3}(x;x')\Big|_{\tau'=\beta_0} \end{aligned} \tag{3.2}$$

($a = 1,2,3$). Semenoff and Weiss [6] assumed that these boundary conditions would also be satisfied by their unperturbed propagators, and this is indeed true. However, the conditions (3.2) are not appropriate for our dissipative propagators, and we now derive the correct version of these conditions.

In order to do perturbation theory, we wish to express the generating functional for unperturbed Green functions

$$Z_0(J_1,J_2,J_3) = \int \mathcal{D}\phi_1\mathcal{D}\phi_2\mathcal{D}\phi_3 \exp\left[\mathrm{i}\tilde{S}_0(\phi_1,\phi_2,\phi_3) + \mathrm{i}\oint \mathrm{d}t\,\mathrm{d}^3x\, J\cdot\phi\right] \tag{3.3}$$

in the form

$$Z_0(J_1,J_2,J_3) = \text{const.}\exp\left[-\frac{1}{2}\int \mathrm{d}t\,\mathrm{d}t'\, J_a(t)g_{ab}(t,t')J_b(t')\right]. \tag{3.4}$$

Supplementing the real-time action (2.13) with the quadratic part of (2.4), we have

$$\begin{aligned} \tilde{S}_0 = \frac{\mathrm{i}}{2}\int_0^T \mathrm{d}t\,[\dot{\phi}_1^2 - \dot{\phi}_2^2 - \beta(t)(\phi_1^2-\phi_2^2) - \gamma(t)(\phi_1\dot{\phi}_2 - \dot{\phi}_1\phi_2) + \mathrm{i}\alpha(t)(\phi_1-\phi_2)^2] \\ + \frac{1}{2}\mathrm{i}\int_0^{\beta_0}\mathrm{d}\tau\,[\dot{\phi}_3^2 + \omega^2\phi_3^2]\,. \end{aligned} \tag{3.5}$$

In the usual fashion, we shift the integration variables according to

$$\phi_a(t) \to \phi_a(t) + \mathrm{i}\int \mathrm{d}t'\, g_{ab}(t,t')J_b(t') \tag{3.6}$$

and seek to eliminate the terms linear in J_a by imposing a condition analogous to (2.10), namely

$$\mathcal{D}(t,\partial/\partial t)\mathbf{g}(t,t') = \mathbf{g}(t,t')\mathcal{D}(t',\overleftarrow{\partial/\partial t'}) = \begin{pmatrix} -\mathrm{i}\delta(t-t') & 0 & 0 \\ 0 & -\mathrm{i}\delta(t-t') & 0 \\ 0 & 0 & -\delta(\tau-\tau') \end{pmatrix} \tag{3.7}$$

where the differential operator is now

$$\mathcal{D} = \begin{pmatrix} \partial^2/\partial t^2 + \beta(t) - \mathrm{i}\alpha(t) & \gamma(t)\partial/\partial t + \frac{1}{2}\dot{\gamma}(t) + \mathrm{i}\alpha & 0 \\ -\gamma(t)\partial/\partial t - \frac{1}{2}\dot{\gamma}(t) + \mathrm{i}\alpha(t) & -\partial^2/\partial t^2 - \beta(t) - \mathrm{i}\alpha(t) & 0 \\ 0 & 0 & \partial^2/\partial\tau^2 - \omega^2 \end{pmatrix}. \tag{3.8}$$

In implementing this procedure, it is necessary to integrate by parts, and boundary conditions on the propagators arise from the requirement that the boundary terms vanish. Taking account of the continuity conditions (3.1) and the corresponding conditions on the fields ($\phi_1(T) = \phi_2(T)$, $\phi_2(0) = \phi_3(0)$ and $\phi_1(0) = \phi_3(\beta_0)$), we obtain

$$\begin{aligned} &\frac{\partial}{\partial t'}g_{a1}(t,t')\Big|_{t'=T} = \frac{\partial}{\partial t'}g_{a2}(t,t')\Big|_{t'=T} \\ &\frac{\partial}{\partial t'}g_{a2}(t,t')\Big|_{t'=0} + \frac{1}{2}\gamma(0)g_{a1}(t,0) = \mathrm{i}\frac{\partial}{\partial\tau'}g_{a3}(t,\tau')\Big|_{\tau'=0} \\ &\frac{\partial}{\partial t'}g_{a1}(t,t')\Big|_{t'=0} + \frac{1}{2}\gamma(0)g_{a2}(t,0) = \mathrm{i}\frac{\partial}{\partial\tau'}g_{a3}(t,\tau')\Big|_{\tau'=\beta_0}. \end{aligned} \tag{3.9}$$

The first of these conditions is automatically satisfied by the real-time propagators described in the previous section. Solving (3.7) for the remaining propagators and applying the conditions (3.1) and (3.9), we find

$$\begin{aligned} g_{33}(\tau,\tau') = {} & h_-(\tau)h_+(\tau')\theta(\tau-\tau') + h_+(\tau)h_-(\tau')\theta(\tau'-\tau) \\ & + (\mathrm{e}^{\beta_0\omega}-1)^{-1}\left[h_-(\tau)h_+(\tau') + h_+(\tau)h_-(\tau')\right] \end{aligned} \tag{3.10}$$

where

$$h_\pm(\tau) = (2\omega)^{-1/2}\mathrm{e}^{\pm\omega\tau} \tag{3.11}$$

and

$$\begin{aligned} g_{13}(t,\tau) = g_{23}(t,\tau) = g_{31}(\tau,t) = g_{32}(\tau,t) \\ = A_1 f(t)h_-(\tau) + A_2 f(t)h_+(\tau) + A_3 f^*(t)h_-(\tau) + A_4 f^*(t)h_+(\tau) \end{aligned} \tag{3.12}$$

with

$$f(t) = (2\Omega(t))^{-1/2}\exp\left[-\tfrac{1}{2}\gamma(0,t) - \mathrm{i}\Omega(0,t)\right] \tag{3.13}$$

the A_i being constants of integration. We note that the imaginary-time propagator (3.10) is identical with that found in the imaginary-time formalism which applies to states of exact thermal equilibrium. Also, the equality of the four mixed-time propagators in (3.12) is required for causality, in the sense that a Green function $G(t_1,\ldots,t_n)$ is independent of the state of the system at times greater than all of the arguments $t_1,\ldots,t_n$.

The values of the constants of integration are conveniently expressed in terms of the quantities

$$N_0 \equiv N(0) = N_1 + N_2 \tag{3.14}$$

$$\dot{N}_0 \equiv \dot{N}(0) = -\gamma_0 N_1 - (\gamma_0 + 2i\Omega_0)N_2 \tag{3.15}$$

where $\gamma_0 \equiv \gamma(0)$ and $\Omega_0 \equiv \Omega(0)$. We find

$$N_0 = \frac{1}{\omega}\left[\Omega_0 + i\frac{\dot{\Omega}_0}{2\Omega_0}\right]\coth\left(\frac{1}{2}\beta_0\omega\right) \tag{3.16}$$

$$\dot{N}_0 = \frac{i}{\omega}\left[\omega^2 - \left(\Omega_0 + i\frac{\dot{\Omega}_0}{2\Omega_0}\right)\left(\Omega_0 + i\frac{\dot{\Omega}_0}{2\Omega_0} - i\gamma_0\right)\right]\coth\left(\frac{1}{2}\beta_0\omega\right) \tag{3.17}$$

with $\dot{\Omega}_0 \equiv \dot{\Omega}(0)$.

Although these results (along with the corresponding values of the A_i, which we do not record here) constitute a formally correct solution of our problem, they are unsatisfactory, because they do not correctly reflect the initial state of equilibrium. Consider, for simplicity, the model with a time-independent mass, which remains in equilibrium. The decay constants γ are non-zero, but we would expect them, along with Ω and N to be time-independent. According to (2.18), we should then have

$$N = \frac{\alpha}{\gamma(\Omega - i\gamma/2)} \tag{3.18}$$

and $\dot{N} = 0$, and no choice of the real quantities α and Ω will make (3.16) and (3.17) consistent with this expectation. To see what has gone wrong, let us temporarily ignore the dissipative counterterm. Our system with time-independent mass will begin and remain in equilibrium so long as the Hamiltonian which appears in the initial density matrix is the same as that which governs the time evolution. Suppose that this is so, but that we foolishly attempt to do perturbation theory by choosing different unperturbed Hamiltonians for the density matrix and the time evolution. Although the whole theory remains in equilibrium, our unperturbed theory obviously will not do so.

Our present difficulty is of the same kind, though not identical, since the dissipative counterterm cannot be represented in terms of a Hamiltonian. To resolve it, we must add to the imaginary-time part of $\bar{S}_0$ a further counterterm which makes the initial equilibrium state of the unperturbed theory consistent with its subsequent time evolution. Naturally, this counterterm will also be subtracted from the interaction, so that the whole theory remains unchanged. The only local counterterm which respects the symmetry (2.14) (or, equivalently, the Hermiticity of the density operator) is of the form

$$\Delta\bar{S}_0 = \int_0^{\beta_0} d\tau \int \frac{d^3k}{(2\pi)^3}\delta_k \phi_{3k}\frac{\partial}{\partial\tau}\phi_{3-k} \tag{3.19}$$

where δ_k is real. Since $\bar{S}$ is homogeneous in imaginary time, we take δ_k to be independent of τ, in which case the integrand in (3.19) is a total derivative. Taking account of the continuity of the fields and suppressing once again the wavevector arguments and integration, we can write

$$\begin{aligned}\Delta\bar{S}_0 &= \tfrac{1}{2}\delta\left[\phi_3^2(\beta_0) - \phi_3^2(0)\right] \\ &= \tfrac{1}{2}\delta\left[\phi_1^2(0) - \phi_2^2(0)\right] \\ &= \tfrac{1}{2}\delta\left[\phi_1(0)\phi_3(\beta_0) - \phi_2(0)\phi_3(0)\right].\end{aligned} \tag{3.20}$$

We see that the new counterterm can be regarded as modifying the differential operator (3.8) with δ-function terms in several equivalent ways. However, it is most straightforward to regard it as leaving $\mathcal{D}$ unchanged but modifying the last two boundary conditions in (3.9), which now become

$$\begin{aligned}\frac{\partial}{\partial t'}g_{a2}(t,t')\bigg|_{t'=0} + \frac{1}{2}\gamma_0 g_{a1}(t,0) &= \mathrm{i}\frac{\partial}{\partial\tau'}g_{a3}(t,\tau')\bigg|_{\tau'=0} - \frac{1}{2}\delta g_{a3}(t,0) \\ \frac{\partial}{\partial t'}g_{a1}(t,t')\bigg|_{t'=0} + \frac{1}{2}\gamma_0 g_{a2}(t,0) &= \mathrm{i}\frac{\partial}{\partial\tau'}g_{a3}(t,\tau')\bigg|_{\tau'=\beta_0} - \frac{1}{2}\delta g_{a3}(t,\beta_0).\end{aligned} \tag{3.21}$$

With these new boundary conditions, the constants of integration (3.14) and (3.15) become

$$N_0 = \frac{1}{\omega}\left[\Omega_0 + \mathrm{i}\frac{\dot{\Omega}_0}{2\Omega_0} - \mathrm{i}\frac{\delta}{2}\right]\coth\left(\frac{1}{2}\beta_0\omega\right) \tag{3.22}$$

$$\dot{N}_0 = \frac{\mathrm{i}}{\omega}\left[\omega^2 - \left(\Omega_0 + \mathrm{i}\frac{\dot{\Omega}_0}{2\Omega_0} - \mathrm{i}\frac{\delta}{2}\right)\left(\Omega_0 + \mathrm{i}\frac{\dot{\Omega}_0}{2\Omega_0} - \mathrm{i}\gamma_0 - \mathrm{i}\frac{\delta}{2}\right)\right]\coth\left(\frac{1}{2}\beta_0\omega\right). \tag{3.23}$$

The functions α, β and γ will now, as before, be determined by a renormalization prescription which partially resums loop corrections to the lowest-order propagators. However, δ will be chosen to ensure that the initial equilibrium state is properly described. From now on, we assume that the solution of (2.17) for $\Omega(t)$ may be chosen so that $\dot{\Omega}_0 = 0$. We see from (3.22) that N_0 will have the correct phase, as in (3.18), if we choose

$$\delta = -\gamma_0 \tag{3.24}$$

and from (3.23) that $\dot{N}_0$ will then vanish if we impose the initial condition

$$\Omega_0 = \left(\omega^2 - \tfrac{1}{4}\gamma_0^2\right)^{1/2} \tag{3.25}$$

on (2.17).

In order to recover (3.18), we would now have to assume that the initial value of $\alpha(t)$ is

$$\alpha_0 = \gamma_0\omega\coth\left(\tfrac{1}{2}\beta_0\omega\right) = \gamma_0\omega(2n_0 + 1) \tag{3.26}$$

where n_0 are the initial occupation numbers (cf equation (3.11) of [9]). This is a constraint on our choice of the renormalization prescription used to determine

$\alpha(t)$. It is satisfied by the prescription adopted in [9], at least at the level of approximation used there. If this constraint is imposed, then we see from (2.18) that $\dot{N} = 0$ at $t = 0$. In the theory with time-independent mass, this will automatically ensure that N remains constant at all times, as it should. In a theory without time-translation invariance, the assumption of an instantaneous initial equilibrium is obviously somewhat artificial, and one might not wish to insist on the conditions (3.24)–(3.26). It is nevertheless important to know that the formalism can be made to describe the case of thermal equilibrium correctly. We record that when these conditions are insisted on, the remaining constants of integration in (3.12) are given by

$$A_1 = \frac{(\Omega_0 - \omega + i\gamma_0/2)}{2\sqrt{(\omega\Omega_0)}(1 - e^{-\beta_0\omega})} = e^{\beta_0\omega} A_4^* \tag{3.27}$$

$$A_2 = \frac{(\Omega_0 + \omega + i\gamma_0/2)}{2\sqrt{(\omega\Omega_0)}(e^{\beta_0\omega} - 1)} = e^{-\beta_0\omega} A_3^*. \tag{3.28}$$

4. Summary and discussion

In this paper we have completed the derivation of a set of propagators suitable for evaluating perturbatively the Green functions of a field theory which is driven away from thermal equilibrium by time-dependent terms in its Hamiltonian. The essential strategy is to use a generalized renormalization procedure to effect a partial resummation of the absorptive parts of loop diagrams so that quasiparticle occupation numbers in the unperturbed propagators evolve with finite relaxation times. We worked in the specific context of a scalar field theory with time-dependent mass, which is assumed to be in thermal equilibrium at some initial time, and further assumed the state to be spatially homogeneous. We summarize here our results for the (3×3) matrix of propagators g_{ab} in the closed-time-path formalism. For $(a, b = 1, 2)$ the propagators have two real-time arguments and, after Fourier transformation on their spatial arguments, depend on a wavevector, which we suppress. They may be written as

$$\mathbf{g}(t,t') = \theta(t-t') \begin{pmatrix} h(t,t') & h^*(t,t') \\ h(t,t') & h^*(t,t') \end{pmatrix} + \theta(t'-t) \begin{pmatrix} h(t',t) & h(t',t) \\ h^*(t',t) & h^*(t',t) \end{pmatrix} \tag{4.1}$$

where

$$h(t,t') = \tfrac{1}{4}\left[\Omega(t)\Omega(t')\right]^{-1/2} \exp\left(-\tfrac{1}{2}\gamma(t',t)\right) \times \left\{\left[1 + N(t')\right] \exp\left(-i\Omega(t',t)\right) + \left[-1 + N^*(t')\right] \exp\left(i\Omega(t',t)\right)\right\} \tag{4.2}$$

and the functions $\gamma(t',t)$ and $\Omega(t',t)$ were defined below (2.16) in terms of time-dependent frequencies $\Omega(t)$ and decay widths $\gamma(t)$. Time-dependent occupation numbers are contained in the functions $N(t)$ displayed in (2.19) where, provided the

conditions (3.24) and (3.25) together with $\dot{\Omega}(0) = 0$ are imposed, the constants of integration are given by

$$N_1 = \left[1 - i\frac{\gamma_0}{2\Omega_0}\right] N_0 \qquad N_2 = i\frac{\gamma_0}{2\Omega_0} N_0 \tag{4.3}$$

with

$$N_0 = \frac{1}{\omega}\left[\Omega_0 + i\frac{\gamma_0}{2}\right] \coth\left(\frac{1}{2}\beta_0\omega\right). \tag{4.4}$$

The remaining propagators are given by (3.10)–(3.13) with the constants of integration (3.27) and (3.28).

It will be observed that the time-dependent occupation numbers are those associated with the earlier of the two time arguments in the propagator, so that no account is taken of the evolution of the state between the two times. This may not be too important, since the propagators decay with the separation of their time arguments, on the same time scale as that which governs the evolution of the occupation numbers. The situation is nevertheless not entirely satisfactory. It arises from our insistence on the use of local counterterms. In principle, one might try to introduce non-local counterterms. The differential equation (3.7) would then become an integro-differential equation, with coefficients $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ which remain to be determined from renormalization conditions. It seems to us that such a formalism would be quite intractable.

The extension of our formalism to spinor and gauge fields, and to more general time-dependent Hamiltonians, should be fairly straightforward, though possibly involving long-winded algebra. It may also be possible to consider more general initial states [7]. In general, one might expect non-equilibrium states to be spatially inhomogeneous. If this inhomogeneity is sufficiently weak, it might be possible to treat it perturbatively, using the homogeneous situation considered here as a lowest-order approximation. However, a complete treatment is likely to be extremely complicated, as is indicated by the discussion of Calzetta *et al* [12], who consider only free fields.

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